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# A CONTACT PROBLEM OF THE THEORY OF CONSOLIDATION FOR A STRIP†

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A plane mixed boundary-value problem of the linear theory of inertialess two-phase consolidation is considered [1]. A strip lying on a smooth undeformable foundation, impermeable to liquid, is under the pressure of a semi-infinite permeable. The material of the solid phase and the liquid are compressible. Using Laplace transformations with respect to time and the space coordinate, the problem is reduced to a Wiener–Hopf equation. The general features of the distribution of the roots of the characteristic equations, corresponding to different homogeneous conditions on the faces of the strip, are investigated. An effective solution is constructed in multiple integrals which converge exponentially with respect to all the variables. The temporal processes of the settling of the punch and the extrusion of the liquid are investigated. © 1999 Elsevier Science Ltd. All rights reserved.

This is the basic problem of the method of piecewise-homogeneous solutions, which can be applied to problems on finite punches, cracks and inclusions in a rectangle, strip or plane. Since the consolidation equations are similar in form to those of associated thermoelasticity, the solution can be applied to the corresponding thermoelastic problems by recalculating the coefficients.

There have not been many analytic solutions of the boundary-value problems of consolidation theory. The basic problems are analysed in [1-7], an approximate solution of the problem of the pressure of a punch on a half-plane saturated with an incompressible liquid is obtained in [8], and the problem of consolidation in a thin layer is solved in [9].

## 1. STATEMENT OF THE PROBLEM

The process of consolidation of a two-phase linearly-deformable isotropic porous medium will be described by the Biot equations [1, 6]

$$G\Delta\mathbf{u}^* + G(1-2\nu)^{-1}\operatorname{grad}\operatorname{div}\mathbf{u}^* - H_1\operatorname{grad}p^* = 0$$
$$k\Delta p^* = H_1\frac{\partial}{\partial t}\operatorname{div}\mathbf{u}^* + (H_2+H_3)\frac{\partial p^*}{\partial t} \tag{1.1}$$

where $\mathbf{u}^*$ is the vector of elastic displacements of the solid phase, $p^*$ is the pore pressure, $t$ is the time, $G$ is the shear modulus, $\nu$ is Poisson's ratio, $k$ is the filtration coefficient, $H_1 = 1-c$, $H_2 = (H_1-f)c_2^{-1}$, $H_3 = fc_3^{-1}$, $c = c_1c_2^{-1}$, $c_1 = \frac{2}{3}G(1+\nu)(1-2\nu)^{-1}$, $f$ is the porosity, $c_1$, $c_2$, $c_3$ are the moduli of bulk compression of the skeleton, the homogeneous isotropic material of the solid phase and the liquid pore respectively, allowing for the gas dissolved in it.

We will investigate plane deformation by writing the equilibrium equations on which system (1.1) is based

$$\frac{\partial\sigma_x^*}{\partial x} + \frac{\partial\tau_{xy}^*}{\partial y} = 0, \quad \sigma_x^* = \sigma_{xs}^* - fp^* \tag{1.2}$$
$$\frac{\partial\tau_{xy}^*}{\partial x} + \frac{\partial\sigma_y^*}{\partial y} = 0, \quad \sigma_y^* = \sigma_{ys}^* - fp^*$$

the generalized Hooke's law for the solid and liquid phases

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$$\sigma_x^* = 2G\left(\frac{\nu}{1-2\nu}\operatorname{div}\mathbf{u}^* + e_x^*\right) - H_1 p^* \tag{1.3}$$

$$\sigma_y^* = 2G\left(\frac{\nu}{1-2\nu}\operatorname{div}\mathbf{u}^* + e_y^*\right) - H_1 p^*, \quad \mathbf{u}^* = (u_x^*, u_y^*) \tag{1.4}$$

$$p^* = \varkappa_1\theta^* + \varkappa_2 \operatorname{div}\mathbf{u}^*, \quad \varkappa_1 = (H_2 + H_3)^{-1}, \quad \varkappa_2 = -H_1\varkappa_1 \tag{1.5}$$

$$\tau_{xy}^* = Ge_{xy}^* \tag{1.6}$$

the Cauchy formulae

$$e_x^* = \partial u_x^* / \partial x, \quad e_y^* = \partial u_y^* / \partial y, \quad e_{xy}^* = \partial u_x^* / \partial y + \partial u_y^* / \partial x$$

the Darcy–Gersevanov law

$$\mathbf{v}^* = -k \operatorname{grad} p^*, \quad \mathbf{v}^* = f(\mathbf{v}_a^* - \partial\mathbf{u}^* / \partial t), \quad \mathbf{v}^* = (\upsilon_x^*, \upsilon_y^*) \tag{1.7}$$

and the equation of continuity

$$\operatorname{div}\mathbf{v}^* = -\partial\theta^* / \partial t, \quad \theta^* = H_1 \operatorname{div}\mathbf{u}^* + (H_2 + H_3)p^* \tag{1.8}$$

The velocity $\mathbf{v}^*$ is equal to the rate of flow of the pore liquid which crosses unit cross-section are of the porous medium relative to the solid phase in unit time, $\mathbf{v}^*$ is in the direction of the normal to the cross-section, $\mathbf{v}_a^*$ is the true mean velocity of the liquid in Euler coordinates, and $\theta^*$ is the change of the original volume of liquid in unit volume of the porous medium after deformation. The stresses $\sigma_{xs}^*$ and $\sigma_{ys}^*$ are taken to mean the forces applied to a skeleton of unit area of the porous medium in the corresponding direction, the stresses in the skeleton are $(1-f)^{-1}$ times greater.

Applying to Eqs (1.1) a Laplace transformation with respect to time and a two-sided Laplace transformation with respect to the $x$ coordinate

$$f(\mu) = \int_0^\infty f^*(t)e^{-\mu t}dt, \quad f^*(t) = \frac{1}{2\pi i}\int_{L_1} f(\mu)e^{\mu t}d\mu \tag{1.9}$$

$$\hat{f}(s) = \int_{-\infty}^{\infty} f(x)e^{-sx}dx, \quad f(x) = \frac{1}{2\pi i}\int_{L_2}\hat{f}(s)e^{sx}ds$$

where $L_j$ is the straight line $\operatorname{Re}\mu = \chi_j$, $\chi_j > 0$ ($j = 1, 2$), we obtain (the symbol ^ is omitted below)

$$\begin{aligned}
&u_x'' + \alpha s^2 u_x + (\beta+1)su_y' - H_1 sp = 0\\
&\alpha u_y'' + s^2 u_y + (\beta+1)su_x' - H_1 p' = 0\\
&p'' + [s^2 - \lambda(H_2 + H_3)]p - \lambda H_1(su_x + u_y') = 0\\
&\alpha = 2(1-\nu)(1-2\nu)^{-1}, \quad \beta = 2\nu(1-2\nu)^{-1}, \quad \lambda = \mu k^{-1}
\end{aligned} \tag{1.10}$$

where $u_x$, $u_y$ and $p$ are the transforms of the displacement vector and pressure and the prime represents the derivative with respect to $y$. In accordance with the "closed system" principle, it is assumed that at the initial time $t = 0$ the liquid does not succeed in leaving the pores, and the initial condition has the form $\theta^*(x, y, 0) \equiv 0$.

We will write the transforms of the stresses

$$\begin{aligned}
&\sigma_x = G(\alpha su_x + \beta u_y') - H_1 p, \quad \sigma_y = G(\beta su_x + \alpha u_y') - H_1 p\\
&\tau = G(u_x' + su_y)
\end{aligned} \tag{1.11}$$

Equations (1.2)–(1.7) lose their asterisks in the transformation, but do not change in any other way and will be applied below without further comment. Equation (1.8) becomes

$$\operatorname{div}\mathbf{v} = -\mu\theta, \quad \theta = \mathrm{H}_1 \operatorname{div}\mathbf{u} + (H_2 + H_3)p \tag{1.12}$$

The general solution of system (1.10) is

$$
\begin{aligned}
&2Gu_x = D_1 \sin sy + D_3(y\eta_1 s \sin sy - \eta_3 \cos sy) + D_5 2Gs \sin qy - \\
&- D_2 \cos sy - D_4(y\eta_1 s \cos sy + \eta_3 \sin sy) - D_6 2Gs \cos qy \\
&2Gu_y = D_1 \cos sy + D_3 y\eta_1 s \cos sy + D_5 2Gq \cos qy + \\
&+ D_2 \sin sy + D_4 y\eta_1 s \sin sy + D_6 2Gq \sin qy \\
&p = D_3 s \cos sy + D_5 \lambda\eta_2 \sin qy + D_4 s \sin sy - D_6 \lambda\eta_2 \cos qy \\
&\eta_j = (\beta + j)\eta + H_1, \quad \eta = (H_2 + H_3)GH_1^{-1}, \quad j = 1, 2, 3 \\
&q^2 = s^2 - \lambda H_4, \quad H_4 = H_2 + H_3 + H_1^2(\alpha G)^{-1}
\end{aligned}
\tag{1.13}
$$

The constants $D_1, \ldots, D_6$ in (1.13) are determined by the boundary conditions for $y = 0, y = 1$, which comprise the elasticity conditions in displacements and stresses for the skeleton and the seepage conditions in pressures and velocities for the pore liquid, as well as the derivatives of these quantities. These could be, for example, the contact conditions for the strip with punches, beams, stringers and Winkler or drainage layers. In the general case these sets of conditions are independent of one another and can be taken in different combinations, generating corresponding characteristic equations $N_m(\mu, s) = 0$ for the eigenvalues $s_k(\mu)$ $(k = 0, \pm 1, \pm 2, \ldots)$, which possess a number of general properties.

We will consider the problem of the pressure on a strip $-\infty < x, < \infty, 0 \leqslant y \leqslant 1$ of a semi-infinite permeable rigid punch $x \geqslant 0, y = 1$ when there is no contact friction on either boundary $y = 0$ or $y = 1$. Assuming that the strip lies on a rigid impermeable foundation, we have the basic conditions

$$
y = 0:\ u_y = 0,\ \ \tau = 0,\ \ \partial p/\partial y = 0; \quad y = 1:\ \tau = 0,\ p = 0 \tag{1.14}
$$

and mixed boundary conditions

$$
y = 1:\ \sigma_y = R(x, \mu),\ \ x < 0; \quad u_y = Q(x, \mu),\ x \geqslant 0 \tag{1.15}
$$

where $R(x, \mu)$, $Q(x, \mu)$ are transforms of the applied load $R^*(x, t)$ and the settlement of the punch $Q^*(x, t)$, respectively.

By satisfying conditions (1.14), we can write (1.13) in the form

$$
\begin{aligned}
&2Gu_x = C(s)[\eta_1\eta_2 s \cos q(y \sin s \sin sy + \cos s \cos sy) + \\
&+ \lambda^{-1} 2Gs \cos s(q \sin q \cos sy - s \sin s \cos qy) - \eta\eta_2 \sin s \cos q \cos sy] \\
&2Gu_y = C(s)[\eta_1\eta_2 s \cos q(y \sin s \cos sy - \cos s \sin sy) - \\
&-\lambda^{-1}\, 2Gsq \cos s(\sin q \sin sy - \sin s \sin qy) - \eta_2^2 \sin s \cos q \sin sy] \\
&p = C(s)\eta_2 s \sin s(\cos q \cos sy - \cos s \cos qy)
\end{aligned}
\tag{1.16}
$$

where $C(s)$ is an arbitrary function.

We will satisfy conditions (1.15). Substituting expressions (1.16) into them, we obtain the two equations

$$
\begin{aligned}
&\sigma^+(s) + \sigma^-(s) = -C(s)sN_1(s)/2 \\
&u^+(s) + u^-(s) = -\eta_2^2 G^{-1} C(s) N_2(s)/2
\end{aligned}
\tag{1.17}
$$

where

$$
\begin{aligned}
&\sigma^+(s) = \int_0^\infty \sigma_y(x, 1)e^{-sx}dx, \quad \sigma^-(s) = \int_{-\infty}^0 R(x)e^{-sx}dx \\
&u^+(s) = \int_0^\infty Q(x)e^{-sx}dx, \quad u^-(s) = \int_{-\infty}^0 u_y(x, 1)e^{-sx}dx
\end{aligned}
\tag{1.18}
$$

The plus and minus superscripts are used to denote that the functions are analytic in the right and left half-planes, respectively.

The characteristic functions of the homogeneous boundary-value problems (1.13)–(1.15) are given by the expressions

$$N_1(s) = N_1(\mu, s) = \eta_1\eta_2 \cos q(2s + \sin 2s) + 4G\lambda^{-1}s \cos s(q \sin q \cos s - s \sin s \cos q)$$
$$N_2(s) = N_2(\mu, s) = \sin^2 s \cos q \tag{1.19}$$

Eliminating the function $C(s)$ from (1.17), we obtain the Wiener–Hopf equation

$$\sigma^+(s) + \sigma^-(s) = K(s)[u^+(s) + u^-(s)], \quad s \in L_2 \tag{1.20}$$
$$K(s) \equiv K(\mu, s) = G\eta_2^{-2} s N_1(s) N_2^{-1}(s)$$

## 2. ANALYSIS OF THE CHARACTERISTIC FUNCTIONS

*Theorem* 1. If $s = i\beta$, $\beta \neq 0$ is a pure imaginary parameter, all the roots of the characteristic equations $N_m(\mu, s) = 0$ $(m = 1, 2, \ldots)$ lie on the ray $\operatorname{Re} \mu \leqslant 0$, $\operatorname{Im} \mu = 0$.

*Proof.* Suppose $\chi = \varepsilon + i\delta$ is the value of the parameter $\mu$, $s = i\beta$ is a simple root of the equation $N_m(\chi, s) = 0$, $u = u(\beta, y, \chi)e^{i\beta}$ is any component of the homogeneous solution. Then the real part $u^r$ and the imaginary part $u^i$ of the function $u$

$$u^r = u^r(\beta, y, \chi) \cos \beta x - u^i(\beta, y, \chi) \sin \beta x$$
$$u^i = u^r(\beta, y, \chi) \sin \beta x + u^i(\beta, y, \chi) \cos \beta x \tag{2.1}$$

do not satisfy Eq. (1.12), which contains the complex parameter $\mu = \chi$, and so are not solutions of the problem. However, they satisfy all the other equations and the homogeneous boundary conditions.

We will now consider the integral over the boundary $\partial\Omega$ of the rectangle $\Omega = \{x, y\colon a \leqslant x \leqslant b, -1 \leqslant y \leqslant 1\}$

$$L^{rr} = -\int_a^b f^{rr}(x, -1)dx + \int_{-1}^1 g^{rr}(b, y)dy + \int_a^b f^{rr}(x, 1)dx - \int_{-1}^1 g^{rr}(a, y)dy$$
$$f^{rr}(x, y) = [C(\tau^r u_x^r + \sigma_y^r u_y^r) + kp^r \partial p^r / \partial y](x, y, \chi) \tag{2.2}$$
$$g^{rr}(x, y) = [C(\sigma_x^r u_x^r + \tau^r u_y^r) + kp^r \partial p^r / \partial x](x, y, \chi)$$

where $C$ is an arbitrary real constant, $b = a + 2\pi\beta^{-1}$.

By virtue of (2.1), $g^{rr}(b, y) = g^{rr}(a, y)$ and therefore the second and fourth integrals of (2.2) cancel each other out. The first and third integrals are equal to zero, since the functions $f(x, \pm 1)$ are sums of products of the real parts of the components of the homogeneous solution, vanishing on the boundaries $y = \pm 1$. Thus, $L^{rr} = 0$.

On the other hand, according to Green's formula

$$\int_{\partial\Omega} -P(x, y)dx + Q(x, y)dy = \iint_{\Omega} \left(\frac{\partial Q}{\partial x} + \frac{\partial P}{\partial y}\right)dxdy$$

if we write $L^{rr}$ in the form of a curvilinear integral over the closed contour $\partial\Omega$, from (2.2) we have

$$C\left[\frac{\partial \tau^r}{\partial y}u_x^r + \tau^r \frac{\partial u_x^r}{\partial y} + \frac{\partial \sigma_y^r}{\partial y}u_y^r + \sigma_y^r \frac{\partial u_y^r}{\partial y} + \frac{\partial \sigma_x^r}{\partial x}u_x^r + \sigma_x^r \frac{\partial u_x^r}{\partial x} + \frac{\partial \tau^r}{\partial x}u_y^r + \tau^r \frac{\partial u_y^r}{\partial x}\right] +$$
$$+k\left[\left(\frac{\partial p^r}{\partial x}\right)^2 + \left(\frac{\partial p^r}{\partial y}\right)^2 + p^r \Delta p^r\right]\Bigg\} dxdy \tag{2.3}$$

We now introduce the vector functions $w^\alpha = (e_x^\alpha, e_y^\alpha, \theta^\alpha, e_{xy}^\alpha)$, $\sigma^\alpha = (\sigma_x^\alpha, \sigma_y^\alpha, p^\alpha, \tau^\alpha)$, the components of which are real or imaginary parts of the homogeneous solution. Let their scalar products be $(\sigma^\alpha, w^\gamma)$, where the superscripts $\alpha$ and $\gamma$ can take the values $r$ and $i$. Substituting (1.5) into (1.3) and (1.4), we obtain $\sigma^\alpha = Aw^\alpha$, where $A$ is a $4 \times 4$ square matrix, $\{a_{ik}\}$

$$a_{11} = a_{22} = 2G(1-\nu)(1-2\nu)^{-1} - H_1\varkappa_2, \quad a_{12} = 2G\nu(1-2\nu)^{-1} - H_1\varkappa_2$$
$$a_{13} = a_{23} = -H_1\varkappa_1, \quad a_{33} = \varkappa_1, \quad a_{14} = a_{24} = a_{34} = 0, \quad a_{44} = G, \quad a_{ki} = a_{ik} \tag{2.4}$$

Since the matrix $A$ is symmetrical $(Aw^\alpha, w^\gamma) = (w^\alpha, Aw^\alpha)$ and, therefore, we have the "Betti formula"

$$(\sigma^\alpha, w^\gamma) = (w^\alpha, \sigma^\gamma) \tag{2.5}$$

From (1.2) the terms in (2.3) with coefficients $u_x^r$ and $u_y^r$ are zero in the sum while the terms with coefficient $\tau^r$ give $\tau^r e_{xy}^r$ in the sum. It follows from Eqs (1.7) and (1.12) that $p^r\Delta p^r = k^{-1}pr(\varepsilon\theta^r - \delta\theta^i)$. Thus $L^{rr}$ can be put in the form

$$L^{rr} = \iint\limits_\Omega \left\{C(\sigma_0^r, w_0^r) + k\left[\left(\frac{\partial p^r}{\partial x}\right)^2 + \left(\frac{\partial p^r}{\partial y}\right)^2 + k^{-1}p^r(\varepsilon\theta^r - \delta\theta^i)\right]\right\}dxdy \tag{2.6}$$

$$\sigma_0^r = (\sigma_x^r, \sigma_y^r, \tau^r), \quad w_0^r = (e_x^r, e_y^r, e_{xy}^r)$$

Since the value of $C$ is arbitrary and $L^{rr} = 0$, we have

$$\iint\limits_\Omega (\sigma_0^r, w_0^r)dxdy = 0 \tag{2.7}$$

We put $C = \varepsilon$. Then from (2.6) and the equation

$$(\sigma^r, w^r) = (\sigma_0^r, w_0^r) + p^r\theta^r \tag{2.8}$$

it follows that

$$L^{rr} = \iint\limits_\Omega \left\{\varepsilon(\sigma^r, w^r) + k\left[\left(\frac{\partial p^r}{\partial x}\right)^2 + \left(\frac{\partial p^r}{\partial y}\right)^2 - \frac{\delta}{k}p^r\theta^i\right]\right\}dxdy \tag{2.9}$$

From (2.4) and (1.5) for any corresponding stresses and strains which do not vanish simultaneously, we have

$$(\sigma, w) = a_{11}(e_x^2 + e_y^2) + 2a_{12}e_xe_y + 2a_{13}(e_x + e_y)\theta + a_{33}\theta^2 + a_{44}e_{xy}^2 =$$
$$= 2G(e_x^2 + e_y^2) + \frac{2G\nu}{1-2\nu}(e_x + e_y)^2 + \varkappa_1[H_1(e_x + e_y) - \theta]^2 + Ge_{xy}^2 > 0$$

Hence from (2.7) and (2.8) it follows that

$$\iint\limits_\Omega (\sigma^r, w^r)dxdy > 0, \quad \iint\limits_\Omega p^r\theta^r dxdy > 0 \tag{2.10}$$

Considering the integral $L^{ii}$ and repeating the argument used for (2.2)–(2.10), we obtain the estimate

$$\iint\limits_\Omega p^i\theta^i dxdy > 0 \tag{2.11}$$

Then, as in (2.9), we derive the equations

$$L^{ri} = \iint\limits_\Omega \left\{\varepsilon(\sigma^r, w^i) + k\left[\frac{\partial p^r}{\partial x}\frac{\partial p^i}{\partial x} + \frac{\partial p^r}{\partial y}\frac{\partial p^i}{\partial y} + \frac{\delta}{k}p^r\theta^r\right]\right\}dxdy = 0 \tag{2.12}$$

$$L^{ir} = \iint\limits_\Omega \left\{\varepsilon(\sigma^i, w^r) + k\left[\frac{\partial p^i}{\partial x}\frac{\partial p^r}{\partial x} + \frac{\partial p^i}{\partial y}\frac{\partial p^r}{\partial y} - \frac{\delta}{k}p^i\theta^i\right]\right\}dxdy = 0 \tag{2.13}$$

Subtracting (2.13) from (2.12) and using formula (2.5), we obtain

$$\frac{\delta}{k}\iint_{\Omega}(p^r\theta^r+p^i\theta^i)dxdy=0$$

which, by virtue of inequalities (2.10) and (2.11), is only possible when $\delta = 0$. Since $L^{\pi} = 0, k > 0$ and $\delta = 0$, it follows from (2.9) and (2.10) that $\varepsilon \leqslant 0$; the equals sign applies when $\partial p^r/\partial x = \partial p^r/\partial y \equiv 0$.

Suppose $s = i\beta$ is an $n$-fold root of the equation $N_m(\chi, s) = 0$. Then the components of one of the $n$ corresponding homogeneous solutions (the residual in the $n$-fold strip (1.9)) have the same form as before $u = u(\beta, y, \chi)e^{i\beta x}$ and obviously are not simultaneously identically equal to zero as a function of $y$ for $s \neq 0$. This case therefore reduces to that of a simple root, which proves the theorem.

*Corollary.* 1. If $\mu = \varepsilon + i\delta, \delta \neq 0$, the functions $N_m(\mu, s)$ have no pure imaginary zeros $s = i\beta, \beta \neq 0$.

*Lemma* 1. The function $K(\mu, s)$ is even in $s$.

*Lemma* 2. The function $K(0) \equiv K(\mu, 0)$ has the following properties:
(a) $K(0) \neq 0$ for any $\mu = i\delta, \delta \neq 0$;
(b) $K(0) > 0$ for real $\mu > 0$.

*Proof.* It is easy to see that

$$\begin{gathered}K(0)=4G\eta_2^{-2}(\eta_1\eta_2-G\lambda^{-1}q^0\operatorname{th}q^0),\quad q^0=\sqrt{\lambda H_4}\\ K(0)\to 2K_0,\quad \lambda\to 0;\quad K(0)\to 4G\eta_1\eta_2^{-1},\quad \lambda\to\infty\\ K_0=2G\eta_2^{-2}(\eta_1\eta_2-GH_4)=G(1-\nu)^{-1}>0;\quad \partial K(0)/\partial\lambda>0\end{gathered}\tag{2.14}$$

This proves the lemma.

*Lemma* 3. For fixed $\mu$ and as $r \to \infty$

$$K(ir)=K_0r+i\chi(r)+O(re^{-2r}),\quad \chi(r)=O(r^{-1})\tag{2.15}$$

*Theorem* 2. For complex values $\mu = \varepsilon + i\delta, \delta \neq 0$, the index $\varkappa$ of the function

$$K_2(s)=K(s)\operatorname{tg}\pi s\,(K_0s)^{-1}\tag{2.16}$$

is equal to zero on the imaginary axis.

*Proof.* By hypothesis, by virtue of Lemmas 1 and 2 and Corollary 1, the function $K_2(s)$ is continuous on the imaginary axis, is even, and has no zeros or poles. It follows that its index, in the sense of the principal Cauchy value, exists and is equal to zero. Taking into account the asymptotic form (2.15), we have

$$\begin{gathered}\ln K_2(ir)=r^{-1}+i\chi(r)+O(re^{-2|r|}),\quad \chi(r)=O(r^{-2}),\quad |r|\to\infty\\ \arg K_2(\pm i\infty)=\lim_{r\to\pm\infty}\operatorname{Im}\ln K_2(ir)=\lim_{r\to\pm\infty}[\operatorname{arctg}\chi(r)r^{-1}]=0\end{gathered}$$

Hence, $\varkappa = 0$ in the usual sense also.

## 3. CONSTRUCTION OF A SOLUTION

We will construct a canonical solution of the homogeneous equation

$$\sigma_0^+(s)=K(s)u_0^-(s),\quad s\in L_2\tag{3.1}$$

By Lemma 3, Corollary 1 and Theorem 2, the function $K(s)$ can be factorized in the familiar way [10]. Thus we obtain

$$u_0^-(s)=\frac{\Gamma(1/2-s)}{\Gamma(1-s)}\exp\left\{-\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\ln K_2(r)}{r-s}dr\right\}\tag{3.2}$$

$$\sigma_0^+(s) = K_0 \frac{\Gamma(1+s)}{\Gamma(1/2+s)} \exp\left\{-\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\ln K_2(r)}{r-s} dr\right\}, \quad s \neq i\beta$$

The integral here, calculated as the principal Cauchy value, converges exponentially. The general solution of Eq. (1.20) can be written in the form suggested by Gakhov in [11].

We will construct a general solution for the case when there is zero applied load and $Q(x, \mu) = -Q(\mu)$ in the simpler form of [12]. From (1.20) we have

$$\sigma^+(s) = K(s)[u^+(s) + u^-(s)], \quad u^+(s) = -Q(\mu)s^{-1} \tag{3.3}$$

where $Q(\mu)$ is the transform of the given time-dependence of the settlement of the punch.

From the condition for finite energy under the edge of the punch, $\sigma_y(x, 1) \sim Ax^{\varphi}$, $\varphi > -1$ as $x \to +0$, we have $\sigma^+(s) \sim A\Gamma(\varphi + 1)s^{-\varphi-1}$ as $s \to \infty$ [12]. Since $\sigma_0^+(s) = O(s^{1/2})$ as $s \to \infty$, by Liouville's theorem we have

$$\sigma^+(s) = -\frac{Q(\mu)\sigma_0^+(s)}{su_0^-(0)}, \quad u_0^-(0) = \sqrt{\frac{K_0}{K(0)}} \tag{3.4}$$

Thus, the solution of the problem is given by formulae (1.9) and (1.16), where

$$C(s) = 2Q(\mu)G\eta_2^{-2} \frac{u_0^-(s)}{su_0^-(0)N_2(s)} \tag{3.5}$$

We will show how a solution can be calculated effectively by performing the inverse transformations (1.9). Consider the pore pressure

$$p^*(x, y, t) = \frac{1}{2\pi i} \int_{L_1} p(x, y)e^{\mu t} d\mu \tag{3.6}$$

$$p(x, y) = \frac{1}{2\pi i} \int_{L_2} C(s)\eta_2 s \sin s(\cos q \cos sy - \cos s \cos qy)e^{sx} ds$$

If $x > 0$, from (1.19) and (3.5) we can write the function $p(x, y)$ in the form

$$p(x, y) = \frac{1}{2\pi i} \frac{2GQ(\mu)}{\eta_2 u_0^-(0)} \int_{L_2} \frac{u_0^-(s)(\cos q \cos sy - \cos s \cos qy)}{\sin s \cos q} e^{sx} ds \tag{3.7}$$

Allowing for the fact that the integrand is meromorphic and closing the contour of integration by a semi-circle in the left half-plane $\operatorname{Re} s \leqslant \chi_2$, we express $p(x, y)$ in the form of a sum of series of the residues at the zeros of the functions $\sin s$ and $\cos q$

$$\begin{aligned} p(x, y) = \frac{2GQ(\mu)}{\eta_2 u_0^-(0)} \Bigg\{ & u_0^-(0)\left(1 - \frac{\operatorname{ch} q^0 y}{\operatorname{ch} q^0}\right) + \sum_{n=-1}^{-\infty} u_0^-(n\pi)\left[(-1)^n \cos n\pi y - \frac{\cos q_n y}{\cos q_n}\right] e^{n\pi x} + \\ & + \sum_{n=-1}^{-\infty} \frac{(-1)^n u_0^-(s_n)(n\pi + \pi/2)\cos s_n \cos(n\pi + \pi/2)y}{s_n \sin s_n} e^{s_n x} \Bigg\} \end{aligned} \tag{3.8}$$

$$q_n = \sqrt{n^2\pi^2 - \lambda H_4}, \quad s_n = -\sqrt{(n\pi + \pi/2)^2 + \lambda H_4}, \quad n = -1, -2, \ldots$$

We will consider two formulations of the problem: (a) when the punch is loaded instantaneously by a given quantity $Q_0$ at the initial time; (b) when the punch is loaded in accordance with a time law which satisfies the conditions $Q^*(0) = 0$, $\dot{Q}^*(t) \geqslant 0$, $Q^*(t) \to Q_0$ as $t \to \infty$. Note that the instantaneous displacement of the punch by a given amount would in fact require infinitely large expenditures of energy, as one can see from the equations of the dynamic theory of consolidation. Then those equations of (1.1) which do not contain inertial terms would prevent us from obtaining a solution which gives an adequate description of the physical process as $t \to 0$.

In the first case $Q(\mu) = Q_0\mu^{-1}$, in the second case with the simplest time law $Q^*(t) = Q_0(1 - e^{-\gamma t})$, where the arbitrary parameter $\gamma > 0$ is used to model the different loading programmes, $Q(\mu) = Q_0\gamma[\mu(\mu + \gamma)]^{-1}$.

We now consider the calculation of $p^*(t)$ using (3.6), (3.8). Changing the order of integration with respect to $\mu$ and of summation of the series in (3.8), we consider the first of the series in (3.8). We close the contour of integration $L_1$ on the left by a contour comprising a quarter-circle of large radius in the second quadrant, the contour $L_3$ including the half-line $\operatorname{Re}\mu = \chi_1$, $\operatorname{Im}\mu = \zeta$, $\zeta > 0$ the straight-line segment $\operatorname{Re}\mu = \chi_1$, $-\zeta \leqslant \operatorname{Im}\mu \leqslant \zeta$, the half-line $\operatorname{Re}\mu \leqslant \chi_1$, $\operatorname{Im}\mu = -\zeta$ and a quarter-circle in the third quadrant. The integrand has no singular points inside this close contour, and the integrals over the arcs of the circles of large radius tend to zero by virtue of Jordan's lemma.

The singular points of the $n$th residue of the integrand in the left half-plane $\operatorname{Re}\mu \leqslant 0$ can be divided into three groups. The first depends on the nature of the applied load and gives simple poles at the point $\mu = 0$ and $\mu = -\gamma$ in this case. The second group are the zeros of $\cos q_n$, which for any $n$ can be computed from the formula

$$\lambda_{nm} = \pi^2 H_4^{-1}[n^2 - (m+1/2)^2], \quad m = 0, \pm 1, \pm 2, \ldots$$

Starting from some $|m| > n$, $\lambda_{nm} < 0$, since there are simple poles on the negative part of the real axis. The singular points of the third group depend on the coefficient $[u_0^-(0)]^{-1}$ which, by (3.4) and (2.14), can be written in the form

$$[u_0^-(0)]^{-1} = 2\sqrt{(1-\nu)\psi(\mu)}, \quad \psi(\mu) = \frac{\eta_1}{\eta_2} - \frac{Gq^0 \operatorname{th} q^0}{\eta_2^2 \lambda} \tag{3.9}$$

Here, on the negative part of the real axis, the radicand vanishes at the points $\mu_m = -t_m^2 k H_4^{-1}$ ($m = 1, 2, \ldots$), where $t_m$ are the roots of the equation

$$\chi t - \operatorname{tg} t = 0, \quad \chi = \eta_1\eta_2(GH_4)^{-1}, \quad \chi \geqslant \alpha \geqslant 2 \tag{3.10}$$

defined by the asymptotic form

$$t_m = \pi m - \pi/2 - 1/(\chi\pi m) + O(1/m^2)$$

These are branch points of the given coefficient, and the corresponding cuts can be assumed to lie entirely on the negative semi-axis.

Thus, in the left half-plane $\operatorname{Re}\mu \leqslant 0$ outside the ray $\operatorname{Re}\mu \leqslant 0$, $\operatorname{Im}\mu = 0$, the given function is analytic, and evaluation of the slowly converging integral over the contour $L_1$ can be replaced by evaluation of the integral of an exponentially decaying function on $L_3$.

For the second series of (3.8), it is extremely effective to evaluate the integrals term-by-term over the contour $L_1$: when $\lambda = ir$ rapid convergence is ensured by the factor $e^{s_n x}$, since

$$s_n = -\sqrt{\rho_n}(\cos\theta_n/2 + i\sin\theta_n/2), \quad \text{where } \rho_n = \sqrt{(n\pi + \pi/2)^4 + r^2 H_4^2}$$

$$\theta_n = \operatorname{arctg}[rH_4(n\pi + \pi/2)^{-2}] \quad \text{and} \quad \theta_n \to \pi/2 \quad \text{as} \quad r \to \infty$$

$p^*(x, y, t)$ and the other components of the solution are calculated in the same way.

## 4. ANALYSIS OF THE SOLUTION

When $x > 0, y = 1$ and $t \to 0$ we will investigate the behaviour of the function $\partial p^*/\partial y$ which defines the flow rate of liquid through a permeable punch at the initial time. By (3.8) we have $\partial p/\partial y \sim Q(\mu)\mu^{1/2}$ on the boundary $y = 1$ as $\mu \to \infty$. It follows from an Abelian-type theorem that the velocity is infinite, $\partial p^*/\partial y \sim t^{1/2}$, $t \to 0$, in the case of an instantaneous load and is equal to zero, $\partial p^*/\partial y \sim t^{1/2}$, $t \to 0$, for an exponential load.

We will determine the normal-stress intensity factor at the edge of the punch as a function of time. From (3.2) and (3.4) we obtain

$$\sigma^+(s) \sim -Q(\mu)\sqrt{K_0 K(0)}\, s^{-1/2} \quad \text{as} \quad s \to \infty$$

Hence, using the relation between the asymptotic forms of the original and the transform, we obtain

$$\sigma_{ys}(x, 1) \sim -Q(\mu)\sqrt{K_0 K(0)}\,(\pi x)^{-1/2} \quad \text{as} \quad x \to +0$$

The normal stresses at the edge of the punch behave with time as given by the expression ($k = 1, 3$)

$$\sigma^*_{ys}(x, 1, t) \sim \frac{K_{\rm I}(t)}{\sqrt{2\pi x}}, \quad K_{\rm I}(t) = -\frac{2\sqrt{2}G}{\sqrt{1-\nu}} \frac{1}{2\pi i} \int_{L_k} Q(\mu)\sqrt{\psi(\mu)}\, e^{\mu t} d\mu \tag{4.1}$$

It can be verified that $\psi(\mu) \to [2(1-\nu)]^{-1}$ as $\mu \to 0$ and, therefore, as $t \to \infty$, according to (4.1), for $k = 1$ we have

$$\sigma^*_{ys}(x, 1) \sim \frac{K_{\rm I}}{\sqrt{2\pi x}}, \quad K_{\rm I} = -\frac{2GQ_0}{1-\nu}$$

which is the same as the asymptotic form of the contact stresses at the edge of the punch in the theory of elasticity corresponding to the problem of consolidation as $t \to \infty$.

Figure 1 shows graphs of the change with time of the stress intensity factors (4.1) for $G = 1000$, $\nu = 1/3$, $c_2 = \infty$ (an incompressible skeletal material) with an instantaneous load for three values of the parameter $\eta$ (the left-hand side and the bottom part of Fig. 1) and with an exponential load for three values of the parameter $\gamma$ and $\eta = 0$ (the top right-hand of Fig. 1); $t_* = kt[G]$ is a dimensionless quantity, $[G]$ is the dimension of $G$, the unit of length is the strip thickness and $K_* \equiv K_{\rm I}(t_*)/K_{\rm I}$. In the first problem the behaviour of the function $\sigma^*_{ys}(x, 1, t)$ as $x \to 0$ for small $t$ is given by the asymptotic formula

$$\sigma^*_{ys}(x, 1, t) \sim \frac{K^0_{\rm I}}{\sqrt{2\pi x}}, \quad K^0_{\rm I} = -\frac{2\sqrt{2}GQ_0}{\sqrt{1-\nu}}\sqrt{\frac{\eta_1}{\eta_2}}$$

which gives the same values of $K_{\rm I}(0)$ as the graphs on the left and bottom of Fig. 1. Comparing the numerical values of the function $K_{\rm I}(t)$ in the figure for $\eta = 0$ and $\gamma = 100$ we see that they agree to three significant figures, even when $t_* = 1.7 \times 10^{-7}$. This justifies using the simpler problem of an instantaneous displacement of the punch, in spite of the infinitely large flow velocity of the liquid as $t \to 0$.

Suppose the punch is pressed into the strip under a uniformly distributed load of intensity $\sigma^*_0(t)$. Then as $x \to \infty$ the contact stresses take the value $\sigma^*_y(x, 1, t) = -\sigma^*_0(t)$, from which we can find the law of the loading of the punch over time $Q^*_0(t)$. In fact, calculating the residue in the strip $s = 0$, from (1.9), (1.11), (1.16), (3.5) and (3.9) we obtain

$$\sigma_y(x, 1, \mu) = 4GQ_0(\mu)\psi(\mu), \quad Q_0(\mu) = [4G\psi(\mu)]^{-1}\sigma_0(\mu)$$

If, in particular, the load is independent of time, $\sigma^*_0(t) = \sigma_0$ then $Q_0(\mu) = [4G\mu\psi(\mu)]^{-1}\sigma_0$. Hence from (1.9), summing the residues in the roots of Eq. (3.10), we obtain a formula for the settlement of the punch

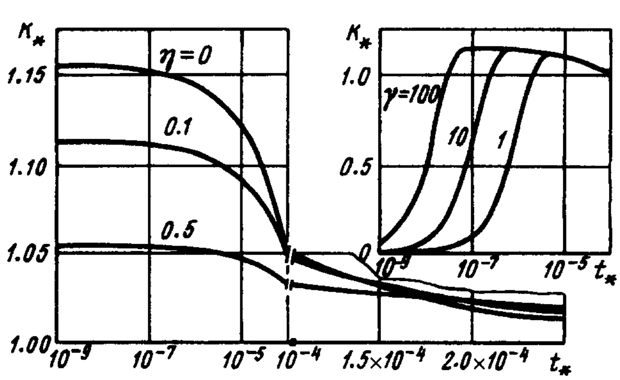


Fig. 1.

$$u_y^*(1,t) = -\frac{\sigma_0}{2G}\left[(1-\nu) - \frac{\alpha\eta_2}{H_1}\sum_{m=1}^{\infty}\frac{\exp(-t_m^2 k H_4^{-1} t)}{\chi^2 t_m^2 + 1 - \chi}\right] \tag{4.2}$$

The total settlement $u_y^s$ is equal to

$$\lim_{t\to\infty} u_y^*(1,t) = -\sigma_0(1-\nu)(2G)^{-1}$$

When $t = 0$ the sum of the series in (4.2) is

$$\sum_{m=1}^{\infty}\frac{1}{\chi^2 t_m^2 + 1 - \chi} = \frac{1}{2\chi(\chi-1)}$$

from which we find the initial settlement

$$u_y^0 = \lim_{t\to 0} u_y^*(1,t) = -\frac{\sigma_0}{4G}\frac{\eta_2}{\eta_1}, \quad \eta_2 \geqslant \eta_1 \geqslant 1$$

If the skeletal material and pore liquid are incompressible, then $\eta_2 = \eta_1 = 1$, $u_y^0/u_y^s = [2(1-\nu)]^{-1}$. If the skeleton itself is incompressible ($\nu \to 1/2$), $u_y^0 = u_y^s$, the punch is immediately totally immersed.

We will now estimate the flow velocity of the liquid at the initial time. As $x \to \infty$ we have

$$v_y^*(x,1,t) = -k\frac{\partial p^*}{\partial y} \sim \frac{\sigma_0\sqrt{H_4 k}}{2\sqrt{\pi}\,\eta_1} t^{-1/2}, \quad t \to 0$$

If the skeletal material and the pore liquid are incompressible, then as $x \to \infty$

$$v_y^*(x,1,t) \sim \frac{\sigma_0\sqrt{(1-2\nu)k}}{2\sqrt{2\pi}\sqrt{(1-\nu)G}} t^{-1/2}, \quad t \to 0$$

It can also be verified that as $x \to \infty$ $\partial p^*/\partial x = 0$, $\tau^* = 0$; $\sigma_x^*$ depends on $y$, but

$$\int_0^1 \sigma_x^* dy = 0$$

Note that as $x \to \infty$ and $t \to 0$ we have $p^*(x,0,t) \to \sigma_0(2\eta_1)^{-1}$, that is, for incompressible skeletal material and pore liquid $p^*(x,0,t) \to \sigma_0/2$.

## REFERENCES

1. BIOT, M. A., General theory of three-dimensional consolidation. *J. Appl. Phys.*, 1941, **12**, 155–165.
2. McNAMEE, J. and GIBSON, R. E., Plane strain and axially symmetric problems of the consolidation of a semi-infinite clay stratus. *Q. J. Mech. Appl. Math.*, 1960, **13**, 210–227.
3. GIBSON, R. E. and McNAMEE, J., A three-dimensional problem of the consolidation of a semi-infinite clay stratum. *Q. J. Mech. Appl. Math.*, 1963, **16**, 115–127.
4. DERSKI, W., Equations of the consolidation theory for the case of a source of fluid. *Bull. Acad. Polon. Sci. Ser. Tech.*, 1965, **13**, 37–43.
5. KERCHMAN, V. I., Problems of consolidation and associated thermoelasticity for a deformable half-space. *Izv. Akad. Nauk SSSR. MTT*, 1976, 45–54.
6. KALININ, N. N. and NULLER, B. M., Transport equations for a fibrous consolidatable material and the near-wall layer effect. *Prikl. Mat. Mekh.*, 1987, **51**, 522–525.
7. GLAGOVSKII, V. B. and NULLER, B. M., Analytic solutions of associated problems of consolidation of a two-phase porous medium for rectangular regions. *Izv. Vsesoyuz. Nauchno-Issled. Inst. Gidromelior. im. B. Ye. Vedeneyeva*, 1991, **223**, 31–40.
8. KERCHMAN, V. I., Contact problem of the theory of consolidation of a water-saturated medium. *Izv. Akad. Nauk SSSR. MTT*, 1974, 3, 102–109.
9. KOVALENKO, Ye. V., The analysis of thin porous coatings. *Prikl. Mat. Mekh.*, 1990, **54**, 469–473.
10. NULLER, B. M., The contact problem for an elastic wedge supported by a rod of equal resistance. *Dokl. Akad. Nauk SSSR*, 1975, **225**, 532–534.
11. GAKHOV, F. D., *Boundary-value Problems*. Nauka, Moscow, 1977.
12. NOBLE, B., *Methods Based on the Wiener–Hopf Technique for the Solution of Partial Differential Equations*. Pergamon Press, Oxford, 1958.

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